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Technical Memorandum

MASKED MATRICES AND POLYNOMIALS

Date: January 16, 1986

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ABSTRACT

Toeplitz matrices are shown to be a special class of a more general class of matrices, called herein "masked matrices", whose elements satisfy a two dimensional linear recursion. An explicit matrix inverse for doubly infinite matrices satisfying a 2X2 mask is derived. The well known explicit inverse of doubly infinite Toeplitz matrices is a special case. It is shown that the inverse of a masked matrix is a masked matrix.

Algebraic properties of masked matrices are also examined. In particular, we introduce an operation by polynomials on masked matrices and we show that this operation induces a decomposition of masked matrices into other masked matrices which are, in a sense, simpler.

I. INTRODUCTION

A doubly infinite Toeplitz matrix is an array of complex numbers

$$T_c = \begin{array}{cl} \left[\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & c_0 & c_{-1} & c_{-2} & c_{-3} & \cdot \\ \cdot & c_1 & c_0 & c_{-1} & c_{-2} & \cdot \\ \cdot & c_2 & c_1 & c_0 & c_{-1} & \cdot \\ \cdot & c_3 & c_2 & c_1 & c_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] & \begin{array}{l} \\ \\ \leftarrow \text{row } 0 \\ \\ \\ \\ \end{array} \\ \begin{array}{cccccc} & & \uparrow & & & \\ & & \text{col } 0 & & & \end{array} & \end{array} \qquad (1)$$

having the property that every entry on any given diagonal is identical. To construct a general Toeplitz matrix, start with the doubly infinite sequence

$$\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots \qquad (2)$$

and then form the associated matrix $T_c = [t_{n,m}]$ with $t_{n,m} = c_{n-m}$. The central question is: Can the system of equations

$$T_c\, x = b \qquad (3)$$

be solved uniquely for the sequence $x = (x_n)$, given the sequence $b = (b_n)$? This question is equivalent to asking when the matrix T_c is invertible.

Define the function $f(\theta)$ to be the Fourier series

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} . \qquad (4)$$

Toeplitz [1] proved that the matrix T_c is invertible if and only if the function $1/f(\theta)$ is essentially bounded. By definition, a function $g(\theta)$ is NOT essentially bounded if and only if each of the sets

$$\left\{\theta : \ |g(\theta)| \geq k\right\} \qquad k=1, 2, \ldots$$

has positive measure. Toeplitz also gave the following explicit formula for T_c^{-1}. The Fourier series expansion

$$\frac{1}{f(\theta)} = \sum_{-\infty}^{\infty} f_n e^{in\theta} \tag{5}$$

gives $T_c^{-1} = [f_{n-m}]$, that is,

$$T_c^{-1} = \begin{array}{ll} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & f_0 & f_{-1} & f_{-2} & f_{-3} & \cdot \\ \cdot & f_1 & f_0 & f_{-1} & f_{-2} & \cdot \\ \cdot & f_2 & f_1 & f_0 & f_{-1} & \cdot \\ \cdot & f_3 & f_2 & f_1 & f_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{row } 0 \\ \\ \\ \\ \end{array} \\ \qquad\qquad\quad \uparrow \text{col } 0 & \end{array} \tag{6}$$

This result is known as the Toeplitz Inversion Theorem [1,2] and will be shown to be a special case of Theorem I.4 below.

The purpose of this paper is to explore extensions of Toeplitz's basic and important results to a more general class of matrices that we call "masked" matrices. In words, a doubly infinite matrix $T = [t_{n,m}]$ is said to be a masked matrix if there exists a complex $(p+1)x(q+1)$ matrix M, with $p \geq 1$ and $q \geq 1$, which is orthogonal to every rectangular submatrix formed from T by eliminating all but p consecutive rows and q consecutive columns. Let $M = [a_{j,k}]$, $j=0, \ldots, p$ and $k=0, \ldots, q$. Then M is a mask for $T = [t_{n,m}]$ if and only if

$$\sum_{j=0}^{p} \sum_{k=0}^{q} \bar{a}_{j,k}\, t_{n+j,m+k} = 0 \text{ for } n, m = 0, \pm 1, \pm 2, \ldots , \tag{7}$$

where $^-$ denote conjugation.

Suppose the matrix T has the 2 x 2 mask

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} . \tag{8}$$

Then, from (7),

$$\sum_{k=0}^{1}\sum_{j=0}^{1} \bar{a}_{jk}\, t_{n+j,\, m+k} = t_{n,m} - t_{n+1,m+1} = 0 \quad \text{for all } n, m.$$

Consequently, any matrix T having the mask (8) must be a Toeplitz matrix. Conversely, if T is Toeplitz, it has the mask (8).

Note that a Toeplitz matrix also has the 3 x 3 mask

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

However, not every matrix satisfying this 3 x 3 mask is necessarily Toeplitz. This points out the fact that a given mask M can be used to characterize a class of matrices. As is easy to see, the class of all matrices satisfying a given mask M forms a linear subspace of the space of all matrices. The dimension of this linear subspace is countably infinite for any mask M.

Consider the general 2 x 2 mask

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{9}$$

where a, b, c, and d are known constants. If the matrix $T = [t_{nm}]$ satisfies this mask, then from (7)

$$\bar{a}\, t_{n,m} + \bar{b}\, t_{n,m+1} + \bar{c}\, t_{n+1,m} + \bar{d}\, t_{n+1,m+1} = 0 . \tag{10}$$

If the mask (9) is the Toeplitz mask (8), then knowledge of one entire column of T suffices to predict all of T. Is this still true for general 2 x 2 masks? The answer is yes, subject to certain conditions, as the following Theorem shows.

<u>Theorem I.1</u>. Given any column of a doubly infinite matrix T, and given that T satisfies the general 2 x 2 mask (9), then T is uniquely determined if and only if both the following conditions hold:

(1) $|a| \neq |c|$ and either $a \neq 0$ or $c \neq 0$
(2) $|b| \neq |d|$ and either $b \neq 0$ or $d \neq 0$.

<u>Proof</u>: Let the given column of T be the m-th column. From (10), the equations for column m+1 are

$$\bar{b}\, t_{n,m+1} + \bar{d}\, t_{n+1,m+1} = -(\bar{a}\, t_{n,m} + \bar{c}\, t_{n+1,m}) \ , \ n=0, \pm 1, \pm 2, \ldots$$

In matrix form this gives the system

$$\begin{array}{r} \\ \\ \\ \text{row } 0 \rightarrow \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \bar{b} & \bar{d} & 0 & 0 & 0 & \cdot \\
\cdot & 0 & \bar{b} & \bar{d} & 0 & 0 & \cdot \\
\cdot & 0 & 0 & \bar{b} & \bar{d} & 0 & \cdot \\
\cdot & 0 & 0 & 0 & \bar{b} & \bar{d} & \cdot \\
\cdot & 0 & 0 & 0 & 0 & \bar{b} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix} \\
\uparrow \\
\text{col } 0
\end{array}
\begin{bmatrix}
\vdots \\ t_{-2,m+1} \\ t_{-1,m+1} \\ t_{0,m+1} \\ t_{1,m+1} \\ t_{2,m+1} \\ \vdots
\end{bmatrix}$$

$$= -
\begin{array}{c}
\begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \bar{a} & \bar{c} & 0 & 0 & 0 & \cdot \\
\cdot & 0 & \bar{a} & \bar{c} & 0 & 0 & \cdot \\
\cdot & 0 & 0 & \bar{a} & \bar{c} & 0 & \cdot \\
\cdot & 0 & 0 & 0 & \bar{a} & \bar{c} & \cdot \\
\cdot & 0 & 0 & 0 & 0 & \bar{a} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix} \\
\uparrow \\
\text{col } 0
\end{array}
\begin{bmatrix}
\vdots \\ t_{-2,m} \\ t_{-1,m} \\ t_{0,m} \\ t_{1,m} \\ t_{2,m} \\ \vdots
\end{bmatrix}
\begin{array}{l} \\ \\ \\ \leftarrow \text{row } 0 \\ \\ \\ \\ \\ \\ \end{array}$$

Both matrices are clearly Toeplitz. By the Toeplitz Inversion Theorem stated above, the matrix on the left has an inverse if and only if

$$f(\theta) = \bar{b} + \bar{d}\, e^{i\theta} \neq 0 \quad \textit{for any } \theta .$$

Since $f(\theta) \neq 0$ if and only if $|b| \neq |d|$ and either $b \neq 0$ or $d \neq 0$, this system can be solved uniquely for column m+1 of T. Similarly, column m+1 of T determines column m if and only if $\bar{a}+\bar{c}e^{i\theta}\neq 0$. By an obvious induction it follows that all of T is uniquely determined. This completes the proof.

Define the doubly infinite Toeplitz matrices M_j by

$$M_j = \begin{array}{cl} \left[\begin{array}{ccccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{0,j} & a_{1,j} & a_{2,j} & \cdots & a_{p,j} & 0 & 0 & \cdot \\ \cdot & 0 & a_{0,j} & a_{1,j} & \cdots & a_{p-1,j} & a_{p,j} & 0 & \cdot \\ \cdot & 0 & 0 & a_{0,j} & \cdots & a_{p-2,j} & a_{p-1,j} & a_{p,j} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] & \begin{array}{l} \\ \\ \leftarrow \text{row } 0 \\ \\ \\ \end{array} \\ \begin{array}{ccccccccc} \quad & \quad & \uparrow & & & & & & \\ & & \text{col } 0 & & & & & & \end{array} & \end{array} \tag{11}$$

By Toeplitz's Inversion Theorem, the matrices M_j are invertible if and only if the functions

$$f_j(\theta) = \sum_{n=0}^{p} a_{n,j}\, e^{in\theta} , \; j = 0, 1, \ldots, q, \tag{12}$$

have no zeros for real values of θ; *i.e.,* $f_j(\theta) \neq 0$ *for any* θ.

<u>Theorem I.2</u>: Let $M = [a_{jk}]$ be a $(p+1) \times (q+1)$ mask for the matrix T. Let C_n denote the n-th column of T, for $n=0, \pm 1, \pm 2, \ldots$. Then

$$\sum_{j=0}^{q} \bar{M}_j\, C_{n+j} = 0 , \; n = 0, \pm 1, \pm 2, \ldots \tag{13}$$

where M_j are the Toeplitz matrices (11).

<u>Proof</u>: Fix n. The definition (7) now gives a system of equations, one each for $m = 0, \pm 1, \pm 2, \ldots$. This system is (13). This concludes the proof.

Corollary I.1. Any q columns of the matrix T determines all of T if none of the functions $f_j(\theta)$, defined by (12), have zeros for real θ.

Proof: The matrices M_j are invertible under the conditions stated. Case 1: suppose that columns 0, ..., q-1 are known. Then

$$C_{n+q} = -\sum_{j=0}^{q-1} \overline{(M_q^{-1} M_j)} \; C_{n+j} \text{ , all n,} \tag{14}$$

which implies that C_{n+q} may be found from C_n, C_{n+1}, ..., C_{n+q-1}. This fills out the right half of T, and similar reasoning fills out the left half. Case 2: Suppose that columns 0, ..., t-1, t+1, ..., q-1 are known. Then (13) can be used to find column t. This reduces the problem to the first case, and so concludes the proof.

We remark that Corollary I.1 is a generalization of Theorem I.1.

An important observation is that the vector

$$E_\theta = (\ldots, e^{-i2\theta}, e^{-i\theta}, 1, e^{i\theta}, e^{i2\theta}, \ldots)^t \tag{15}$$

is an eigenvector of every matrix M_j, defined by (11), and that the eigenvalue corresponding to E_θ is $f_j(\theta)$, defined by (12). Thus

$$M_j \; E_\theta = f_j(\theta) \; E_\theta. \tag{16}$$

Because M_j is Toeplitz, we also have

$$E_\theta^t \, M_j = \bar{f}_j(\theta) \; E_\theta^t \; . \tag{17}$$

The proof of this fact is a straightforward verification.

Now, multiplying (13) on the left by $\bar{E}_\theta^{\,t}$ gives

$$0 = (\bar{E}_\theta)^t \sum_{j=0}^{q} \bar{M}_j \; C_{n+j}$$

$$= \sum_{j=0}^{q} \overline{(E_\Theta^t M_j)} \; C_{n+j}$$

$$= \sum_{j=0}^{q} \overline{\bar{f}_j(\Theta) \; E_\Theta^t} \; C_{n+j}$$

$$= \sum_{j=0}^{q} f_j(\Theta) \left(\bar{E}_\Theta^t \; C_{n+j} \right)$$

$$= \sum_{j=0}^{q} f_j(\Theta) \; \bar{F}_{n+j}(\Theta)$$

where we have defined the functions

$$F_n(\Theta) = \bar{E}_\Theta{}^t \; C_n$$

$$= \sum_{k=-\infty}^{\infty} t_{k,n} e^{ik\Theta}. \tag{18}$$

We stress that the existence of $F_n(\Theta)$ defined by the Fourier series (18) is an assumption. Although the Corollary I.1 proves that each column of T exists uniquely, we have not shown that each column gives a convergent Fourier series (18). We have proved the following result.

Theorem I.3: If M is a (p+1) x (q+1) mask for the matrix T, and if every function $F_n(\Theta)$ defined by (18) is convergent, then

$$\sum_{j=0}^{q} f_j(\Theta) \; \bar{F}_{n+j}(\Theta) = 0, \; n=0, \pm 1, \pm 2, \ldots, \tag{19}$$

where $f_j(\theta)$ is defined by (12).

Corollary I.2: If $p = q = 1$,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$f_0(\theta) = a + c\, e^{i\theta} \neq 0$$
$$f_1(\theta) = b + d\, e^{i\theta} \neq 0$$

then

$$F_n(\theta) = (-1)^n \overline{\left(\frac{a + c\, e^{i\theta}}{b + d\, e^{i\theta}}\right)}^{\,n} F_0(\theta), \quad n=0, \pm 1, \pm 2, \ldots \tag{20}$$

Proof: From (19), for $n \geq 0$,

$$f_0(\theta)\, \bar{F}_n(\theta) + f_1(\theta)\, \bar{F}_{n+1}(\theta) = 0$$

so that

$$F_{n+1}(\theta) = -\overline{\left(\frac{f_0(\theta)}{f_1(\theta)}\right)} F_n(\theta)$$

which implies half of (20). Similarly,

$$f_0(\theta)\, \bar{F}_{-n-1}(\theta) + f_1(\theta)\, \bar{F}_{-n}(\theta) = 0$$

or

$$F_{-(n+1)}(\theta) = -\overline{\left(\frac{f_0(\theta)}{f_1(\theta)}\right)}^{-1} F_{-n}(\theta).$$

This completes the proof.

Corollary I.3: Suppose there does not exist any value of θ such that $F_n(\theta)=0$ for all n. Define the trigonometric polynominal

$$\sum_{j=0}^{q}\sum_{k=0}^{p} a_{kj}\, e^{i(k\theta+j\phi)} = D(\theta,\phi). \tag{21}$$

Then, for every θ there exists ϕ such that $D(\theta,\phi) = 0$.

<u>Proof</u>: Fix θ. From (19) we get the Toeplitz system

$$\begin{array}{rc}
 & \left[\begin{array}{cccccccccc}
\cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & f_0(\theta) & f_1(\theta) & f_2(\theta) & \ldots & f_q(\theta) & 0 & 0 & \cdot \\
\text{row } 0 \rightarrow & \cdot & 0 & f_0(\theta) & f_1(\theta) & \ldots & f_{q-1}(\theta) & f_q(\theta) & 0 & \cdot \\
 & \cdot & 0 & 0 & f_0(\theta) & \ldots & f_{q-2}(\theta) & f_{q-1}(\theta) & f_q(\theta) & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\end{array}
\begin{bmatrix} \vdots \\ F_{-1}(\theta) \\ F_0(\theta) \\ F_1(\theta) \\ \vdots \end{bmatrix} = 0$$

$$\begin{array}{c} \uparrow \\ \text{col } 0 \end{array}$$

By Toeplitz Inversion Theorem this system has a unique solution if and only if $D(\theta,\phi)$, as a function of ϕ, does not vanish. But if $D(\theta,\phi) \neq 0$ for any ϕ, then $F_n(\theta) = 0$ for all n, since the right hand side of this Toeplitz system is identically zero, contradicting the hypothesis. Thus, there exists ϕ such that $D(\theta,\phi) = 0$. This completes the proof.

We can solve formally the general system $T\,x = b$ for x when the matrix T has a 2 x 2 mask, that is, $p = q = 1$. The next theorem gives an explicit matrix inverse for this case. The problem for other values of p and q is more difficult.

<u>Theorem I.4</u>: Let $p = q = 1$, and let the mask M be as in Corollary I.2 for the matrix $T = [t_{n,m}]$. Suppose that $T^{-1} = [s_{n,m}]$ exists and that, for every m, the function

$$\sum_{n=-\infty}^{\infty} s_{n,m}\, z^n$$

is defined on the unit circle Γ and on the set

$$L^{-1}(\Gamma) = \left\{ z\varepsilon\ C : z = L^{-1}(w),\ w\varepsilon\Gamma \right\}$$

where

$$L(z) = -\frac{a + b\,z}{c + d\,z}, \quad ad - bc \neq 0. \tag{21}$$

Suppose that

(1) $F_0(z) = \sum_{n=-\infty}^{\infty} t_{n,0}\, z^n$ exists and is defined on Γ and $L(\Gamma)$

(2) $1/F_0(L(e^{i\theta}))$ is essentially bounded for all real θ

(3) $\frac{1}{F_0(z)} = \sum_{k=-\infty}^{\infty} f_k\, z^k$ exists and is defined on the set $L(\Gamma)$.

Then, the general entry $s_{n,m}$ of T^{-1} is given explicitly by

$$s_{n,m} = \sum_{k=-\infty}^{\infty} (-1)^{m+k} f_k\, I(m+k,n) \tag{22}$$

where, for all integers μ and ν, we define

$$I(\mu,\nu) = \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{a\, z + b}{c\, z + d}\right)^{\mu} \frac{d\, z}{z^{\nu + 1}} \, . \tag{23}$$

<u>Proof</u>: Consider the general system $T\, x = b$ or, equivalently,

$$\sum_{k=-\infty}^{\infty} t_{n,k}\, x_k = b_n \, , \; n=0, \pm 1, \ldots \quad . \tag{24}$$

Let z be a complex variable. Multiply both sides of (24) by z^n and sum over n to get

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} t_{n,k}\, x_k\, z^n = \sum_{n=-\infty}^{\infty} b_n\, z^n \equiv B(z) \, . \tag{25}$$

All series are assumed to be absolutely summable, so interchange the order of summation to get

$$B(z) = \sum_{k=-\infty}^{\infty} x_k \; F_k(z)$$

where (see (18))

$$F_k(z) = \sum_{n=-\infty}^{\infty} t_{n,k} \; z^n, \; k=0, \pm 1, \; \ldots \tag{26}$$

By Corollary I.2, we have

$$B(z) = F_0(z) \sum_{k=-\infty}^{\infty} x_k \; \overline{\left(- \frac{a + c\ z}{b + d\ z}\right)}^k . \tag{27}$$

Because ad - bc $\neq$ 0, $L^{-1}(x)$ exists and equals

$$L^{-1}(z) = - \frac{a + c\ z}{b + d\ z}. \tag{28}$$

Hence, from (27), we write

$$B(z) = F_0(z) \sum_{k=-\infty}^{\infty} x_k \; \overline{\left[L^{-1}\ (z)\right]}^{\,k}$$

$$= F_0(z) \; X \left(\overline{\left[L^{-1}\ (z)\right]}\right) . \tag{29}$$

where

$$X(z) = \sum_{k=-\infty}^{\infty} x_k \; z^k \; . \tag{30}$$

We will use $X(z)$ only for sequences (x_k) which are the columns of T^{-1}; by hypothesis, then, we have that $X(z)$ is defined on the sets $L^{-1}(\Gamma)$ and Γ. Since $B(z)$ and $F_0(z)$ are assumed to be defined on the set $L(\Gamma)$, we have

$$X(z) = \frac{B(L(\bar{z}))}{F_0(L(\bar{z}))} . \tag{31}$$

Using (30) and (31) gives the n-th coefficient of $X(z)$ as

$$x_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{B(L(e^{-i\theta}))}{F_0(L(e^{-i\theta}))} e^{-in\theta} \, d\theta.$$

By assumption

$$\frac{1}{F(L(e^{i\theta}))} = \sum_{k=-\infty}^{\infty} f_k \, (L(e^{i\theta}))^k$$

is essentially bounded for all θ, so

$$x_n = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f_k \int_{-\pi}^{\pi} B(L(e^{-i\theta})) \, (L(e^{-i\theta}))^k \, e^{-in\theta} \, d\theta .$$

To find the general entry $s_{n,m}$ of T^{-1}, we take $B(z) = z^m$, corresponding to choosing the vector b to have all zero components, except the m-th component which is 1. With this choice of b, we have $x_n = s_{n,m}$ and

$$s_{n,m} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f_k \int_{-\pi}^{\pi} \left(- \frac{a + b\, e^{-i\theta}}{c + d\, e^{-i\theta}} \right)^{m+k} e^{-in\theta} \, d\theta$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (-1)^{m+k} f_k \int_{-\pi}^{\pi} \left(\frac{a\, e^{i\theta} + b}{c\, e^{i\theta} + d} \right)^{m+k} e^{-in\theta} \, d\theta$$

$$= \sum_{k=-\infty}^{\infty} (-1)^{m+k} f_k \, I(m + k, n).$$

This completes the proof.

As an interesting corollary to this result, we are able to obtain a mask for the inverse of a masked matrix,

Corollary I.5: Let T and T^{-1} be as in theorem I.4. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a mask for T, then $\tilde{M} = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix}$ is a mask for T^{-1}.

Proof: Before we begin this proof, it is worthwhile to note that conjugates appear in $\tilde{M}$ but disappear in the proof since the definition of a mask requires us to take conjugates before applying the mask. According to theorem I.4, we may represent $s_{n,m}$ as

$$s_{n,m} = \sum_{k=-\infty}^{\infty} (-1)^{m+k} f_k \, I(m+k, n).$$

Applying the mask $\tilde{M}$ gives

$$a \, s_{n,m} + c \, s_{n,m+1} + b \, s_{n+1,m} + d \, s_{n+1,m+1}$$

$$= \sum_{k=-\infty}^{\infty} \Big\{ (-1)^{m+k} f_k \; a \, I(m+k, n) + (-1)^{m+1+k} f_k \; c \, I(m+1+k, n)$$

$$+ (-1)^{m+k} f_k \; b \, I(m+k, n+1) + (-1)^{m+1+k} f_k \; d \, I(m+1+k, n+1) \Big\}$$

$$= \sum_{k=-\infty}^{\infty} (-1)^{m+k} f_k \Big\{ a \, I(m+k, n) - c \, I(m+1+k, n) + b \, I(m+k, n+1)$$

$$- d \, I(m+k+1, n+1) \Big\}.$$

The sum within the braces reduces to an integral over Γ which has *the* integrand

$$a\left(\frac{az+b}{cz+d}\right)^{m+k}\frac{1}{z^{n+1}} - c\left(\frac{az+b}{cz+d}\right)^{m+k+1}\frac{1}{z^{n+1}} + b\left(\frac{az+b}{cz+d}\right)^{m+k}\frac{1}{z^{n+2}} - d\left(\frac{az+b}{cz+d}\right)^{m+k+1}\frac{1}{z^{n+2}}$$

$$= \left(\frac{az+b}{cz+d}\right)^{m+k}\frac{1}{z^{n+1}}\left[a - c\left(\frac{az+b}{cz+d}\right) + b\,\frac{1}{z} - d\left(\frac{az+b}{cz+d}\right)\frac{1}{z}\right]$$

Now adding fractions, we find the numerator of the *bracket* is

$$a\,z(cz+d) - c(az+b)z + b(cz+d) - d(az+b)$$

$$= (ac - ca)z^2 + (ad - bc + bc - da)z + bd - db$$

$$= 0\ .$$

Thus, the mask of the inverse is the Hermitian transpose of the original mask.

Toeplitz's Inversion Theorem is also a corollary of this result. With the mask (8), we have

$$I(\mu, \upsilon) = \frac{1}{2\pi i}\int_{\Gamma} (-1)^{\mu}\, z^{\mu-\upsilon-1}\, dz$$

$$= (-1)^{\mu}\, \delta_{\mu,\upsilon}$$

where δ is Kronecker's delta. Hence, from (22),

$$s_{n,m} = \sum_{k} (-1)^{m+k} f_k \left\{(-1)^{m+k}\, \delta_{m+k,n}\right\}$$

$$= f_{n-m}$$

exactly as given by (6).

II. Masked Matrices and Polynomials

In this section we will explore a connection which exists between masked matrices and polynomials in two variables over the complex numbers. When we have developed this connection we will be able to exploit the well known arithmetic of polynomials to further our understanding of masked matrices. In order to do this, however, we need to establish some notation and to fix some ideas. Throughout this section, C will denote the complex numbers and Z will denote the integers. First, we state formally our definition of a masked matrix.

<u>Definition II.1</u>: Let $T = [t_{n,m}]_{n,m \in Z}$ be a doubly infinite matrix over the complex numbers. Let $M = [a_{k,j}]$, $k = 0, \ldots, p$, $j = 0, \ldots, q$ be a matrix of complex numbers. We say M is a <u>matrix mask</u> for T if and only if

$$\sum_{j=0}^{q} \sum_{k=0}^{p} \bar{a}_{k,j}\, t_{r+k,s+j} = 0 \quad \text{for } r,s = 0, \pm 1, \pm 2, \ldots$$

where $^{-}$ denotes complex conjugation.

Intuitively, in order to decide whether or not M is a mask for T, we place M on T with the element $a_{0,0}$ on top of t_{rs}, a_{01} on top of $t_{r+1,s}$, etcetera. Then we multiply $\bar{a}_{k,j}$ by $t_{r+k,s+j}$ and add the products. If that sum is zero for every choice of r and s, then M is a mask for T.

Let us now consider a more general action of M on any complex doubly infinite matrix. From now on, DM denotes the complex vector space of all doubly infinite matrices. If M is a p + 1 by q + 1 matrix over C, we define a linear operator $\mathcal{J}_M$: DM $\rightarrow$ DM as follows. If T ε DM, then the (r,s) entry of $\mathcal{J}_M(T)$ is given by

$$\sum_{k=0}^{q} \sum_{j=0}^{p} \bar{a}_{k,j}\, t_{r+k,s+j} \quad \text{for all } r,s = 0, \pm 1, \pm 2, \ldots$$

Intuitively, in order to calculate the (r,s) entry of $\mathcal{J}_M(T)$, we place M on T with the element $a_{0,0}$ on top of $t_{r,s}$, $a_{0,1}$ on top of $t_{r,s+1}$ etcetera. Then we multiply $\bar{a}_{k,j}$ by $t_{r+k,s+j}$ and add the products. We state some of the properties of $\mathcal{J}_M$ in the following theorem whose proof is obvious.

<u>Theorem II.1</u>: Let M be a p + 1 by q + 1 matrix over C and define $\mathcal{T}_M$ as above. Then $\mathcal{T}_M$ is a linear transformation on DM. If T ε DM, then M is a matrix mask for T if and only if T is in the kernel of $\mathcal{T}_M$.

As an example, consider the Toeplitz mask

$$M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If T is any element of DM, then the (r,s) entry of $\mathcal{T}_M(T)$ is $t_{r+1,s+1} - t_{r,s}$.

Some facts about masked matrices follow immediately from the theorem. For example, since the kernel of $\mathcal{T}_M$ is a subspace, if M is a matrix mask for both T_1 and T_2 then M must be a matrix mask for $T_1 + T_2$. Furthermore, $\mathcal{T}_M$ is zero if and only if M is an entirely zero matrix.

Some obvious properties of $\mathcal{T}_M$ are worth keeping in mind. For example, $\mathcal{T}_{[0\ 1]}$ merely shifts every doubly infinite matrix one column to the left while $\mathcal{T}_{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$ shifts every doubly infinite matrix one row upward. The linear transformation $\mathcal{T}_{[1]}$ is the identity. Finally, $\mathcal{T}_{M_2} = \mathcal{T}_{M_2}$ if and only if M_1 and M_2 are obtained from some matrix M_3 by the concatenation of rows of zeroes at the bottom and columns of zeroes on the right.

Now we shall adopt a new point of view which will allow us to bring in some more mathematical machinery. Suppose we have a p + 1 by q + 1 matrix M = $[a_{k,j}]$, k = 0, ..., p, j = 0, ..., q. Then it is possible to interpret the k-th row of M as a polynomial $f_k(x)$ where the coefficient of x^j is a_{kj}. We may then interpret the entire matrix as a polynomial in two variables x and y where the coefficient of y^k is $f_k(x)$.

For example, if M is the Toeplitz mask, $f_0(x) = -1 \bullet x^0 + 0 \bullet x = -1$ and $f_1(x) = 0 \bullet x^0 + 1 \bullet x = x$, so the entire polynomial is xy - 1. Similarly, if we consider the Hankel mask,

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

we get the polynomial y - x. The row matrix [0 1] corresponds to x while the column matrix $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponds to y.

Conversely, if we are given a polynomial in x and y, we get a matrix as follows. If the degree of the polynomial in x is q and the degree of the polynomial in y is p, we will have a p + 1 by q + 1 matrix. The entry in row

r and column s in the coefficient of $x^s y^r$. For example, the polynomial $1 + x + xy$ gives rise to the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We should note that the correspondence between matrices and polynomials is not quite one to one. That is, if we start with a matrix, derive a polynomial and then derive a matrix, we might not end with the original matrix. That is because the original matrix might have had a column of zeroes at its right or a row of zeroes at its bottom. But either of these cases would be cut off in the transition to a polynomial and then back to a matrix.

There is an obvious relationship between addition of matrices and addition of polynomials under our correspondence provided we are willing to juggle sizes of matrices and degrees of polynomials. However, there is no obvious correspondence between polynomial multiplication and the usual matrix multiplication. We would be very surprised if there were since polynomial multiplication is commutative and matrix multiplication is not. However, we can define a convolution of matrices by using the multiplication of polynomials.

<u>Definition II.2</u>: Let M_1 be a $p + 1$ by $q + 1$ matrix and let M_2 be an $r + 1$ by $s + 1$ matrix. Then $M_1 * M_2$ is a $p + r + 1$ by $q + s + 1$ matrix.

The (k,j) entry of $M_1 * M_2$ is the sum of all products $m^1_{ab}\, m^2_{cd}$

where $a + c = k$, $b + d = j$ and the superscripts 1 and 2 refer to entries in

M_1 and M_2 respectively. $M_1 * M_2$ is the convolution of M_1 and M_2.

The polynomial corresponding to $M_1 * M_2$ is equal to the product of the polynomials of M_1 and M_2. Thus, we see that convolving M_1 and M_2 corresponds to turning M_1 and M_2 into polynomials, multiplying those polynomials as usual and then returning a matrix according to the conventions we established before.

For example, if

$$M_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then we multiply $xy - 1$ by $y - x$ to get $xy^2 + (-x^2 - 1)\, y + x$. Thus

$$M_1 * M_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Using the fact that convolution corresponds to polynomial multiplication we see at once that it is commutative, associative and distributes over matrix addition. Furthermore, the 1 x 1 matrix [1] acts as a unit for *.

It is interesting to note the effect of convolving M with [0 1]. Since the latter corresponds to the polynomial x, M*[0 1] is the same as M except that a column of zeroes has been added on the left. Similarly, $M*\begin{bmatrix}0\\1\end{bmatrix}$ is the same as M except that a row of zeroes has been added on top.

Now we can use the correspondence between matrices and polynomials to introduce polynomial operators onto the space of doubly infinite matrices. From now on, we use the notation C[x] to denote polynomials in one variable over the complex numbers and C[x,y] to denote polynomials in two variables.

<u>Definition II.3</u>: Let $\phi(x,y) = \sum_{k=0}^{q} \sum_{j=0}^{p} a_{j,k}\, x^k y^j \in C[x,y]$. Let T be a doubly infinite matrix over C. Define $\phi(x,y)T$ by the formula

$$(\phi(x,y)T)_{r,s} = \sum_{k=0}^{q} \sum_{j=0}^{p} \bar{a}_{j,k}\, t_{r+j\ s+k}$$

<u>Definition II.4</u>: Let $\phi(x,y) \in C[x,y]$ and let $T \in DM$. We say ϕ is a polynomial mask for T if and only if $\phi(x,y)T = 0$ where 0 is the zero matrix.

The theorem below tells us that the operation of polynomials on doubly infinite matrices actually makes sense.

<u>Theorem II.2</u>: Using the above multiplication, DM is a module over C[x,y].

<u>Proof</u>: The only module property which is not at once obvious is the associative law. So let $\phi(x,y) = \sum_{k=0}^{q} \sum_{j=0}^{p} a_{j,k}\, x^k y^j$ and $\Psi(x,y) = \sum_{m=0}^{s} \sum_{n=0}^{r} b_{n,m}\, x^m y^n$ be the elements of C[x,y] and let $T \in DM$. Then

$$(\emptyset(x,y)(\psi(x,y)T)_{uv} = \sum_{k=0}^{q}\sum_{j=0}^{p} \overline{a_{j,k}}\ (\ (x,y)T)_{u+j,v+k}$$

$$= \sum_{k=0}^{q}\sum_{j=0}^{p} \overline{a_{j,k}}\left(\sum_{m=0}^{s}\sum_{n=0}^{r} \overline{b_{n,m}}\ t_{u+j+n,v+k+m}\right)$$

$$= \sum_{k=0}^{q}\sum_{j=0}^{p}\sum_{m=0}^{s}\sum_{n=0}^{r} \overline{a_{j,k}}\ \overline{b_{n,m}}\ t_{u+j+n,v+k+m}$$

We can change indices on the last sum and rewrite it as:

$$\sum_{\mu=0}^{q+s}\sum_{\nu=0}^{p+r}\left(\sum_{(j,n)}\ \sum_{(k,m)} \overline{a_{j,k}}\ \overline{b_{n,m}}\right) t_{u+\mu,v+\nu}$$

where the sum in the middle is taken over all pairs (j,n) and (k,m) where $j + k = \mu$, $n + m = \nu$. But the sum in the middle is the coefficient of $x^{\mu}y^{\nu}$ in $\emptyset(x,y)\ \psi(x,y)$ and that is all we need to complete the proof.

We can use this theorem to prove some elementary facts about masks.

Proposition II.3: Let $\emptyset_1(x,y)$, $\emptyset_2(x,y) \in C[x,y]$ and T_1, $T_2 \in DM$. Then, if $\emptyset_1(x,y)$ and $\emptyset_2(x,y)$ are both polynomial masks for T_1, so is $\emptyset_1(x,y) + \emptyset_2(x,y)$. If $\emptyset_1(x,y)$ is a polynomial mask for T_1 and $\emptyset_2(x,y)$ is a polynomial mask for T_2, then $\emptyset_1(x,y)\ \emptyset_2(x,y)$ is a polynomial mask for $T_1 + T_2$. Finally, if $\emptyset_1(x,y)$ is a polynomial mask for T_1, then $\emptyset_1(x,y)\ \emptyset_2(x,y)$ is also a polynomial mask for T_1.

Proof: These are just straightforward calculations. For example, if $\emptyset_1(x,y)$ is a polynomial mask for T_1 and $\emptyset_2(x,y)$ is a polynomial mask for T_2 then

$$\begin{aligned}\emptyset_1(x,y)\ \emptyset_2(x,y)\ (T_1 + T_2) &= \emptyset_1(x,y)\ \emptyset_2(x,y)T_1 + \emptyset_1(x,y)\ \emptyset_2(x,y)T_2 \\ &= \emptyset_2(x,y)\ \emptyset_1(x,y)T_1 + 0 \\ &= 0 + 0 \\ &= 0\ .\end{aligned}$$

The other facts follow just as easily.

Let us examine some examples using the above theorems. A Toeplitz matrix satisfies $xy - 1$. By the above theorem, it also satisfies $\phi(x,y)(xy - 1)$ for any $\phi(x,y)$. If $\phi(x,y) = xy - 1$ for example, we find that a Toeplitz matrix also satisfies $x^2y^2 - 2xy + 1$. That is, the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is also a mask for a Toeplitz matrix. However, not every matrix which satisfies this mask is necessarily Toeplitz.

As another example, consider the Toeplitz mask $xy - 1$ and the Hankel mask $y - x$. Then, according to the above theorem, a polynomial mask for a Toeplitz matrix plus a Hankel matrix is given by $(xy - 1)(y - x) = xy^2 - x^2y - y + x$. That is,

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

is a Toeplitz-plus-Hankel mask.

Finally, suppose we have a matrix which is both Toeplitz and Hankel. Then it must satisfy both $xy - 1$ and $y - x$. But, using the theorem, it must also satisfy $xy - 1 - x(y - x) = x^2 - 1$. So it must satisfy the matrix mask $[-1\ 0\ 1]$. Therefore, any row must look like

$$\ldots\ a\ b\ a\ b\ a\ b\ \ldots$$

By the Toeplitz property, any other row must be a shift of this row. Thus any matrix which is both Toeplitz and Hankel looks like

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & a & b & a & b & \cdot \\ \cdot & b & a & b & a & b & a & \cdot \\ \cdot & a & b & a & b & a & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

We formalize the above remarks in a theorem whose proof is so routine we omit it.

Theorem II.4: Let $T \in DM$. The set of all polynomial masks for T is an ideal in $C[x,y]$. Conversely, if $\phi(x,y) \in C[x,y]$, the set of all $T \in DM$ for which $\phi(x,y)$ is a mask is a submodule of DM.

III. Decomposition of Masked Matrices

Part of Theorem II.3 may be restated by saying that a polynomial mask for a sum is the product of the polynomial masks for the separate summands. Therefore, it is natural to ask the following question. Suppose $\phi_1(x,y)\ \phi_2(x,y)$ is a polynomial mask for T. Are there matrices T_1 and T_2 such that $\phi_1(x,y)$ is a mask for T_1, $\phi_2(x,y)$ is a mask for T_2 and $T = T_1 + T_2$? In general, the answer to this question is no. However, if we insist that $\phi_1(x,y)$ and $\phi_2(x,y)$ be relatively prime, then the question does indeed have an affirmative answer. In particular, the main theorems of this section are theorems III.13, III.14 and III.17 which formally state the manner in which the decompositions take place.

In order to prove the decomposition we shall need some auxiliary results concerning masks for row vectors. Therefore, we shall let DV stand for the set of doubly infinite row vectors over the complex numbers. If $v \in DV$, we shall picture it as $(\ldots, v_{-1}, v_0, v_1, \ldots)$. If $f(x) = \sum_{k=0}^{n} a_k x^k \in C[x]$, we define $f(x)v$ by letting the j-th entry be $\sum_{k=0}^{n} \bar{a}_k v_{j+k}$. Then all of the properties that held true for doubly infinite matrices over $C[x,y]$ are easily proved for doubly infinite vectors over $C[x]$. In particular, we say f(x) is a polynomial mask for v if $f(x)v = \underline{0}$.

If $a \in C$ is non-zero, we let $\underline{a}$ be the doubly infinite vector whose k-th entry is a^k. We let $\underline{0}$ be the doubly infinite vector whose entries are all zero. For non-zero a and $j \geq 0$, let $\underline{a}^{(j)}$ be the doubly infinite vector whose k-th entry is

$$k(k-1)\ldots(k-j+1)(\bar{a})^{k-j}$$

Thus $\underline{a}^{(0)} = \underline{a}$, and $a^{(1)} = (\ldots, -2(\bar{a})^{-3}, -(\bar{a})^{-2}, 0, 1, 2\bar{a}, \ldots)$. In general, the k-th entry of $\underline{a}^{(j)}$ is the j-th derivative of $\bar{a}^{i}$ with respect to $\bar{a}$.

We use the $\underline{a}$'s heavily in the following lemmas.

Lemma III.1: Let $v \in DV$ and $a \in C$. Then $(x - a)v = \underline{0}$ if and only if v is a scalar multiple of $\underline{a}$.

Proof: If $a = 0$, the theorem is obvious and we omit the proof. Likewise, if v is $c\underline{a}$ we omit the calculation that $(x - a)\, c\underline{a} = 0$. However, suppose $(x - a)v = \underline{0}$. Then the equation says that $v_{i+1} = \bar{a}v_i$. Thus

$$v_1 = \bar{a}v_0\ ,\ v_2 = \bar{a}^2 v_0,\ \text{etc. Furthermore, } v_{-1} = (\bar{a})^{-1}v_0,\ v_{-2} = (\bar{a})^{-2}v_0,\ \ldots$$

That is, $v = v_0\, \underline{a}$.

Lemma III.2: Let $a \in C$ be non-zero. Then $(x - a)\underline{a}^{(j)} = j\underline{a}^{(j-1)}$.

Proof: Let $v = (x - a)\underline{a}^j$. Then

$$V_k = (k+1)k\ \ldots\ ((k+1)-(j+1))\ (\bar{a})^{k+1-j} - \bar{a}\ k\ (k-1)\ \ldots\ (k-j+1)(\bar{a})^{k-j}$$

$$= k(k-1)\ \ldots\ (k-j+2)\ (\bar{a})^{k-j+1}\ ((k+1)-(k-j+1))$$

$$= j\ k(k-1)\ \ldots\ (k-j+2)\ (\bar{a})^{k-j+1}$$

and this is the k-th entry of $j\ \underline{a}^{(j-1)}$

We will use these to prove:

Theorem III.3: Let $a \in C$ be non-zero and let $v \in DV$. $(x - a)^k v = 0$ if and only if v is a linear combination of $\underline{a}$, $\underline{a}^{(1)}$, ..., $\underline{a}^{(k-1)}$.

Proof: We use induction on k. We have already proved the theorem true for $k = 1$. Suppose we know it is true for k and we wish to prove it for $k + 1$.

If $(x - a)^{k+1}v = 0$, then $(x - a)^k v$ is in the kernel of $x - a$. Thus $(x - a)^k v = c\underline{a}$ by theorem III.1. By theorem III.2, we see that $(x - a)^k (c/k!)\ \underline{a}^{(k)} = c\underline{a}$. Thus $v = (c/k!)\ \underline{a}^{(k)} + w$ where $(x - a)^k w = 0$. But, using the induction hypothesis, w is a linear combination of $\underline{a}$, ..., $\underline{a}^{(k-1)}$. That tells us that v is a linear combination of $\underline{a}$, $\underline{a}^{(1)}$, ..., $\underline{a}^{(k)}$ and we are done.

We have now successfully determined the kernel of every polynomial operator of the form $(x - a)^k$. We would now like to determine the kernel of every polynomial operator. Since C is algebraically closed, we know that every polynomial in $C[x]$ factors uniquely as a product of a constant times a set of polynomials of the form $(x - a)^k$. We should thus be able to apply theorem III.3 to find the kernel of all operators. We will, however, need one more lemma.

Lemma III.4: Let $f(x) = g(x)h(x) \in C[x]$ where $g(x)$ and $h(x)$ have no roots in common. Let $v \in DV$. Then $f(x)v = \underline{0}$ if and only if there are v_1 and v_2 in DV with $g(x)v_1 = \underline{0}$, $h(x)v_2 = \underline{0}$ and $v = v_1 + v_2$.

Proof: The "if" part of the theorem is obvious. For the only if part, we suppose $f(x)v = \underline{0}$. We now recall that the Euclidean algorithm tells us

that if $g(x)$ and $h(x)$ are relatively prime, there are polynomials $r(x), s(x) \varepsilon C[x]$ with the property that $r(x)g(x) + s(x)h(x) = 1$. Thus $r(x)g(x)v + s(x)h(x)v = v$. Let $v_2 = r(x)g(x)v$. Then $h(x)v_2 = h(x)r(x)g(x)v = r(x)g(x)h(x)v = r(x)f(x)v = \underline{0}$. Similarly, if we let $v_1 = s(x)h(x)v$, then $g(x)v_1 = \underline{0}$.

<u>Theorem III.5</u>: Suppose $p(x) \varepsilon C[x]$ where $p(x) = a_0(x-a_1)^{k_1} \ldots (x-a_n)^{k_n}$. If $v \varepsilon DV$, then $p(x)v = \underline{0}$ if and only if v is a linear combination of $\underline{a}_1, \ldots, \underline{a}_1^{(k_1-1)}, \ldots, \underline{a}_n, \ldots, \underline{a}_n^{(k_n-1)}$.

<u>Proof</u>: We use induction on n. The case n=1 has been covered in theorem III.3. So assume we know the theorem is true for n and we will prove it for n+1. In that case, $p(x) = a_0(x-a_1)^{k_1} \ldots (x-a_n)^{k_n}(x-a_{n+1})^{k_{n+1}}$. From theorem III.4 we know that $p(x)v = \underline{0}$ if and only if $v = v_1 + v_2$ where $(x-a_{n+1})^{k_{n+1}} v_1 = \underline{0}$ and $(x-a_1)^{k_1} \ldots (x-a_n)^{k_n} v_2 = \underline{0}$. But the induction step tells us that v_2 is a linear combination of $\underline{a}_1, \ldots, \underline{a}_1^{(k_1-1)}, \ldots, \underline{a}_n, \ldots, \underline{a}_n^{(k_n-1)}$ and theorem III.3 tells us that v_1 is a linear combination of $\underline{a}_{n+1}, \ldots, \underline{a}_{n+1}^{(k_{n+1}-1)}$ and we are done.

Let us now proceed from doubly infinite vectors to doubly infinite matrices where we shall make use of the preceding theorems. Before we go too far, we should make a few observations concerning polynomials in C[x] considered as operators on DM. In the first place, we note that if $f(x) \varepsilon C[x]$, then $f(x)$ operates on elements of DM in a row-wise manner. That is, $f(x)$ operates on one row at a time.

Furthermore, if $a \varepsilon C$, then we may "stack" scalar multiples of $\underline{a}$ to form a doubly infinite matrix. We shall denote such a stacked matrix by $\underline{c} \cdot \underline{a}$ where we think of $\underline{c}$ as a doubly infinite column vector with elements from C. (Intuitively speaking, $\underline{c}$ is the column vector formed by picking out the elements in column zero of $\underline{c} \cdot \underline{a}$.) Similar remarks apply to the row vectors $\underline{a}^{(j)}$. With these ideas in mind, we have the following theorem whose proof is obvious.

<u>Theorem III.6</u>: Let $p(x) \varepsilon C[x]$ and let $T \varepsilon DM$ be such that $p(x)T = 0$. Then T is a linear combination of doubly infinite matrices of the form $\underline{c} \cdot \underline{a}^{(j)}$ where $(x-a)^{j+1}$ is a divisor of $p(x)$.

We shall also need the following technical lemmas.

Lemma III.7: Let $T \in DM$ and $a \in C$. Then there is a $U \in DM$ with the property that $(x-a)U = T$.

Proof: If $a=0$, we need only shift the columns of T one unit to the right to obtain U. If a is not zero, then we only need to consider the action of x-a on a typical row. For row i, the equation $(x-a)U = T$ states that $u_{i,j+1} - \bar{a}\, u_{i,j} = t_{i,j}$. If $j=0$, set $u_{1,0} = 0$. Now solve this set of equations recursively.

Corollary III.8: Let $T \in DM$ and let $p(x) \in C[x]$. Then there is a $U \in DM$ with $p(x)U = T$.

Proof: All we need to do is factor p(x) into factors of the form (x-a) and use the above theorem several times.

The next lemma is somewhat different in tone from the preceding. With a little thought, it is easy to see that it concerns solutions of Toeplitz systems of equations.

Lemma III.9: Let $c_1, \ldots, c_n \in C$ be not all zero. Let $\underline{d}$ be any doubly infinite column vector over C. Then there is another doubly infinite column vector $\underline{w}$ over C with the property

$$c_1 w_k + c_2 w_{k+1} + \ldots + c_n w_{k+n-1} = d_k$$

for each $k \in Z$.

Proof: We may assume without loss of generality that c_1 and c_n are not zero. First, pick any numbers $w_0, \ldots, w_{n-1}$ which solve the above equation for i=0. Now let k=1. We now have an equation which involves $c_1, \ldots, c_n, w_1, \ldots, w_{n-1}$ and d_1 which are quantities we already know. The only other variable is w_n for which we may solve. In this way, we can solve for any w_k simply by sliding the quantities $c_1, \ldots, c_n$ along the vector w and picking up one more unknown at a time.

We shall use lemma III.9 in the proof of lemma III.10. In algebraic terms, lemma III.10 states that certain matrices are divisible by certain polynomials.

Theorem III.10: Let $a \in C$ be non-zero. Let $T \in DM$ be such that $(x-a)^k T = \underline{0}$. Let $\phi(x,y) \in C[x,y]$ be such that x-a does not divide $\phi(x,y)$.

Then there is a $U \in DM$ such that $T = \phi(x,y)U$ and $(x-a)^k U = \underline{0}$.

Proof: We shall write $\phi(x,y) = f_0(x) + f_1(x,y) + \ldots + f_q(x)y^q$. Then, the non-divisibility condition states that not all of $f_0(a), \ldots, f_n(a)$ are zero.

We now proceed by induction. Suppose k=1. Then (x-a)T=0. Thus, T = $\underline{c}$. $\underline{a}$ for some column vector $\underline{c}$. Consider the equation $\phi(x,y)\underline{d} \cdot \underline{a} = \underline{c} \cdot \underline{a}$ where the vector $\underline{d}$ consists of unknowns. Because of the way $\phi(x,y)$ operates on doubly infinite matrices, we see that the (m,n) entry of $\phi(x,y)\underline{d} \cdot \underline{a}$ is

$$\bar{a}_{0,0}\, d_m \bar{a}^n + \bar{a}_{0,1}\, d_m \bar{a}^{n+1} + \dots + \bar{a}_{0,q}\, d_m \bar{a}^{n+q}$$

$$+ \bar{a}_{1,0}\, d_{m+1} \bar{a}^n + \bar{a}_{1,1}\, d_{m+1}\, \bar{a}^{n+1} + \dots + \bar{a}_{1,q}\, d_{m+1}\, \bar{a}^{n+q}$$

$$\vdots$$

$$+ \bar{a}_{p,0}\, d_{m+p} \bar{a}^n + \bar{a}_{p,1}\, d_{m+p}\, \bar{a}^{n+1} + \dots + \bar{a}_{p,q}\, d_{m+p}\, \bar{a}^{n+q}$$

$$= \overline{f_0(a)}\, d_m\, \bar{a}^n + \overline{f_1(a)}\, d_{m+1}\, \bar{a}^n + \dots + \overline{f_p(a)}\, d_{m+p}\, \bar{a}^n$$

This entry should be equal to $c_m \bar{a}^n$. Thus, we only need to solve the system of equations

$$\overline{f_0(a)}\, d_m + \overline{f_1(a)}\, d_{m+1} + \dots + \overline{f_p(a)}\, d_{m+p} = c_m$$

for each m. But we know we can do this using the fact that not all of the $f_0(a)$, ..., $f_p(a)$ are zero and applying the previous lemma.

Now we have a matrix $\underline{d}$. $\underline{a}$ with $\phi(x,y)$ d . a = c . a = T and (x-a) d.a = 0. Suppose we know the theorem is true for k and we wish to prove it for k+1. Consider $(x-a)^{k+1}T = 0$. Then $(x-a)^k((x-a)T) = 0$. By induction, $(x-a)T = \phi(x,y)U$ and $(x-a)^kU = 0$. Pick a V with (x-a)V = U. (Lemma III.7 says this can be done). Then $(x-a)(T-\phi(x,y)V) = 0$. Thus, $T - \phi(x,y)V = \phi(x,y)W$ where (x-a)W = 0. But then $T = \phi(x,y)(V+W)$ and obviously, $(x-a)^{k+1}(V+W) = 0$.

<u>Corollary III.11</u>: Let T $\in$ DM and suppose that $p(x) \in C[x]$ with p(x)T = 0. Let $\phi(x,y) \in C[x,y]$ with the property that if a is a root of p(x), then (x-a) does not divide $\phi(x,y)$. Then there is a U $\in$ DM such that $T = \phi(x,y)U$ and p(x)U = 0.

<u>Proof</u>: Apply theorem III.10 once for each root of p(x) to get the desired U. We can tie the above together to get the following.

<u>Theorem III.12</u>: Let $p(x) \in C[x]$ and $\phi(x,y) \in C[x,y]$. Suppose that if a is a root of $p(x)$, then $(x-a)$ does not divide $\phi(x,y)$. If $T \in DM$ has the property that $p(x)\phi(x,y)T = 0$, then $T = U+V$ where $U, V \in DM$ and $\phi(x,y)V = 0$ and $p(x)U = 0$.

<u>Proof</u>: From Corollary III.11, we see that there is a U with $\phi(x,y)T = \phi(x,y)U$ and $p(x)U = 0$. If we let $V = T-U$, we have the desired decomposition.

We can now mount an assault on our main theorem: Suppose we have $T \in DM$ and $\phi(x,y)$ and (x,y) in $C[x,y]$. Suppose, further, that $\phi(x,y)$ and $\psi(x,y)$ are relatively prime and $\phi(x,y)\psi(x,y)T = 0$. We would like to decompose T as $U+V$ where $\phi(x,y)U = 0$ and $\Psi(x,y)V = 0$. The above theorems show that, if one of the polynomials involves only x, the decomposition occurs. Therefore, from now on we may assume that both $\phi(x,y)$ and $\psi(x,y)$ involve y non-trivially and that they have no factors which are polynomials in x alone.

We shall need some facts concerning polynomials in $C[x,y]$ and those same polynomials in $C(x)[y]$. This second ring consists of polynomials in y whose coefficients are rational functions in x. If follows from Gauss's lemma that irreducible polynomials in $C[x,y]$ remain irreducible when considered as elements of $C(x)[y]$. Thus, if $\phi(x,y)$ and $\psi(x,y)$ are relatively prime in $C[x,y]$, they remain relatively prime in $C(x)[y]$. In $C(x)[y]$ we may find elements $\gamma_0(x,y)$ and $\delta_0(x,y)$ with the property that $\gamma_0(x,y)\phi(x,y) + \delta_0(x,y)\psi(x,y) = 1$. If we clear denominators, we now have the equation $\gamma_1(x,y)\phi(x,y) + \delta_1(x,y)\psi(x,y) = q(x)$ in $C[x,y]$. These facts will be useful in the following proof.

<u>Theorem III.13</u>: Let $T \in DM$ and suppose $\phi(x,y)\psi(x,y)T = 0$ where $\phi(x,y)$, $\Psi(x,y) \in C[x,y]$ are relatively prime. Suppose further that neither $\phi(x,y)$ nor $\Psi(x,y)$ has a nontrivial factor which involves only x. Then $T = U+V$ where $\phi(x,y)U = 0$ and $\Psi(x,y)V = 0$.

<u>Proof</u>: We make use of the equation $\gamma_1(x,y)\phi[x,y) + \delta_1(x,y)\psi(x,y) = q(x)$. From Corollary III.8, we know there is $W \in DM$ with $q(x)W = T$. Then $\gamma_1(x,y)\phi(x,y)W + \delta_1(x,y)\psi(x,y)W = T$. From the definition of W, we see that $q(x)\phi(x,y)\psi(x,y)W = 0$. If we apply Gauss's lemma, we see that, if a is a root of $q(x)$, then $x-a$ does not divide $\phi(x,y)\psi(x,y)$. Thus, we may find W_1 and W_2 with the properties that $W = W_1 + W_2$, $\phi(x,y)\psi(x,y)W_1 = 0$ and $q(x)W_2 = 0$. If we let $U = \delta_1(x,y)\psi(x,y)\, W_1$ and $V = \gamma_1(x,y)\phi(x,y)\, W_1$, we see that $\phi(x,y)U = 0$ and $\psi(x,y)V = 0$. Furthermore, $U+V = \delta_1(x,y)\psi(x,y)W_1 + \gamma_1(x,y)\phi(x,y)W_1$

$= q(x)W_1$

$= q(x)(W_1+W_2)$

$= q(x)W$

$= T$

The above theorem allows us to make use of unique factorization in C[x,y] to get a more general decomposition.

<u>Theorem III.14</u>: Let $\psi(x,y) \in C[x,y]$ and $T \in DM$. Suppose $\psi(x,y)$ has the unique factorization $\psi(x,y) = \phi_1(x,y)^{e_1} \ldots \phi_k(x,y)^{e_k}$ in C[x,y] and furthermore, suppose $\psi(x,y)T = 0$. Then there are $U_1, \ldots, U_k \in DM$ such that $T = U_1 + \ldots + U_k$ and $\phi_j(x,y)^{e_j} U_j = 0$ for $j = 1, \ldots, k$

<u>Proof</u>: We need only use a simple induction on the number of factors. Obviously, the converses of theorem III.13 and theorem III.14 holds also. In closing this section, we should point out that theorem III.14 holds also for masked matrices of finite size. However, we need to modify the definition of a mask so that it makes sense for m by n matrices.

<u>Definition III.1</u>: Let $T = [t_{r,s}]$, r=1, ..., m, s=1, ..., n be an m by n matrix over C and let $M = [a_{k,j}]$, k=0, ..., p, j=0, ..., q be a p+1 by q+1 matrix over C. We say M is a matrix mask for T if and only if

$$\sum_{j=0}^{q} \sum_{k=0}^{p} \bar{a}_{k,j} t_{r+k,s+j} = 0$$

for all r and s with $r+p \leq m$ and $s+q \leq n$.

The point is that we must ensure that the mask M does not go outside the bounds of the matrix T.

Now suppose we are given a matrix T of finite size which satisfies a mask M. The theorem below tells us that we may enlarge T to another matrix T^1 such that T^1 also satisfies M.

<u>Theorem III.15</u>: Let T be an m by n matrix over C which satisfies $M = [a_{i,j}]$ which is a p by q matrix mask. Then there is a column m vector v such that T^1, the m by n+1 matrix obtained by adjoining v to the right of T, also satisfies M.

<u>Proof</u>: We picture the construction of v geometrically by adjoining a column of unknowns to the right of T. Now we think of placing the first q-1 columns of M on top of the last q-1 columns of T so that the first row of M lies over the first row of T and the last column of M lies over the column of unknowns. Suppose that k is the index of the highest numbered row such that a_{kq} is not zero. Assign any values to the first k unknowns so that M becomes a mask for the submatrix over which it currently lies. Next, move M down one row and assign a value to the unknown numbered k+1 so that M is a gain a mask for the submatrix over which it lies. Continue in this manner assigning values to all of the unknowns. Then v is the vector obtained from those unknowns.

Corollary III.16: Let T be an m by n matrix which satisfies a p by q matrix mask M. Then there is a doubly infinite matrix T^1 such that T is a submatrix of T^1 and T^1 satisfies M.

Proof: We use theorem III.15 to add rows and columns to T in order to get larger matrices. A simple induction then allows us to conclude that T^1 exists.

We conclude this section with our main statement about finite matrices.

Theorem III.17: Let T be an m by n matrix over C and let $\psi(x,y) \varepsilon C[x,y]$. Suppose $\psi(x,y)$ has the unique factorization $\phi_1(x,y)^{e_1} \ldots \phi_k(x,y)^{e_k}$ in C[x,y]. Then T satisfies the polynomial mask $\psi(x,y)$ if and only if there exist m by n matrices $U_1, \ldots, U_k$ such that U_j satisfies the polynomial mask $\phi_j(x,y)^{e_j}$ and $T = U_1 + \ldots + U_k$.

Proof: Suppose T satisfies $\psi(x,y)$. Then there is us a doubly infinite matrix T^1 which contains T as a submatrix and which satisfies $\psi(x,y)$. Decompose T^1 as a sum of doubly infinite matrices $U_1^1, \ldots, U_k^1$ such that U_j^1 satisfies $\phi_j(x,y)^{e_j}$. Then let U be the submatrix of U^1 which corresponds to the submatrix T of T^1. Obviously U_j satisfies $\phi_j(x,y)_j^{e_j}$ and $T = U_1 + \ldots + U_j$. The converse is similar.

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